above consists in the use of the key property of the set of resonance tori of the unperturbed problem.

The author thanks V.I. Arnol'd and Iu. A. Arkhangel'skii for attention and advice.

REFERENCES

1. Deprit, A., Study of the free rotation of a solid body around a fixed point, using the phase plane. Mekhanika (collection of translations), № 2, 1968.
2. Siegel, K. L., Lectures on Celestial Mechanics. Moscow, Izd. Inostr. Lit., 1959.
3. Golubev, V. V., Lectures on the Integration of the Equations of Motion of a Heavy Solid Body Around a Fixed Point. Moscow, Gostekhizdat, 1953.
4. Sadov, Iu. A., The action-angle variables in the Euler-Poinsot problem. PMM Vol. 34, № 5, 1970.
5. Arnol'd, V. I., Proof of A. N. Kolmogorov's theorem on the preservation of conditionally-periodic motions under a small variation of the Hamiltonian function. Uspekhi Matem. Nauk, Vol. 18, № 5, 1963.
6. Poincaré, H., New Methods in Celestial Mechanics. Collected Works, Vol. 1, Moscow, "Nauka", 1971.
7. Arnold, V. I. and Avez, A., Problèmes ergodiques de la méchanique classique. Paris, Gauthier-Villars, 1967.
8. Poincaré, H., On the three-body problem and on the equations of dynamics. Collected Works, Vol. 2, Moscow, "Nauka", 1972.

Translated by N. H. C.

UDC 531.36

ON THE STABILITY OF MOTION IN CRITICAL CASES

PMM Vol. 39, № 3, 1975, pp. 415-421
A. S. OZIRANER
(Moscow)
(Received October 29, 1974)

We examine a class of functions of higher than first order in smallness in the equations of perturbed motion, for which the stability problem in critical cases is completely solved in a linear approximation. More precisely, we give a generalization of Malkin's theorem on the singular case of several zero roots to the case when the characteristic equation has pure imaginary roots. We consider the instability question.

1. Let us consider a system of differential equations of perturbed motion (1.1), where the functions Y_i and X_s satisfy conditions (1.2), (1.3)

$$y_i^{\bullet} = q_{i1}x_1 + \ldots + q_{in}x_n + Y_i(t, \mathbf{x}, \mathbf{y}), \quad i = 1,\ldots,k \tag{1.1}$$
$$x_s^{\bullet} = p_{s1}x_1 + \ldots + p_{sn}x_n + X_s(t, \mathbf{x}, \mathbf{y}), \quad s = 1,\ldots,n$$

$$Y_i(t, 0, \mathbf{y}) \equiv X_s(t, 0, \mathbf{y}) \equiv 0 \tag{1.2}$$

$$\frac{\|\mathbf{Y}(t, \mathbf{x}, \mathbf{y})\| + \|\mathbf{X}(t, \mathbf{x}, \mathbf{y})\|}{\|\mathbf{x}\|} \underset{t\geqslant 0}{\rightrightarrows} 0 \quad \text{as} \quad \|\mathbf{x}\| + \|\mathbf{y}\| \to 0 \tag{1.3}$$

while the roots of the equation

$$\det \| p_{ij} - \lambda\delta_{ij} \| = 0 \tag{1.4}$$

have negative real parts. Malkin's theorem [1, 2] is valid under these conditions: the unperturbed motion $y_i = x_s = 0$ is Liapunov-stable and exponentially-asymptotically x-stable. It is not difficult to see that the stability of the unperturbed motion of system (1.1) is of the same nature as the stability of the trivial solution of the linear system obtained from (1.1) by discarding the functions Y_i and X_s. Thus, Malkin's theorem can be stated in the following manner: if the zero solution of the linear system (1.1) with $Y_i \equiv X_s \equiv 0$ is Liapunov-stable and exponentially-asymptotically x-stable, then the stability of the zero solution of the original system (1.1) is of the same nature for every function Y_i, X_s satisfying conditions (1.2), (1.3).

The characteristic equation for the linear system (1.1) with $Y_i \equiv X_s \equiv 0$ has k zero roots with prime elementary divisors and n roots with negative real parts. In this connection we pose the following problem: is Malkin's theorem preserved when the characteristic equation of the first-approximation system has pure imaginary roots in addition to zero roots. It happens that this question has an affirmative answer both in the case of stability as well as in the case of instability.

2. Let the equations of perturbed motion have the form

$$\mathbf{y}^{\boldsymbol{\cdot}} = Q\mathbf{x} + R\mathbf{y} + \mathbf{Y}(t, \mathbf{x}, \mathbf{y}), \quad \mathbf{x}^{\boldsymbol{\cdot}} = P\mathbf{x} + \mathbf{X}(t, \mathbf{x}, \mathbf{y}) \tag{2.1}$$

where $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{y} \in \mathbf{R}^k$ and P, Q, R are constant matrices of the appropriate orders. The linear-approximation equations for system (2.1) are

$$\mathbf{y}^{\boldsymbol{\cdot}} = Q\mathbf{x} + R\mathbf{y}, \quad \mathbf{x}^{\boldsymbol{\cdot}} = P\mathbf{x} \tag{2.2}$$

T h e o r e m 1. Assume that the zero solution of system (2.2) is Liapunov-stable and exponentially-asymptotically x-stable. Then the unperturbed motion of system (2.1) is Liapunov-stable and exponentially-asymptotically x-stable with any functions $\mathbf{Y}$, $\mathbf{X}$ satisfying conditions (1.2) and (1.3).

N o t e. As we see from the theorem's statement, the real parts of the roots of Eq. (1.4) are negative, while the roots of the equation

$$\det \| R - \lambda E \| = 0 \tag{2.3}$$

may have negative real parts, may vanish, or be pure imaginary. In the last two cases a number of solution groups equal to the root multiplicity are associated with multiple roots, and resonance relations are admissible between imaginary roots.

P r o o f [1, 2]. By the property of the roots of Eq. (1.4) there exists a positive-definite quadratic form $V(\mathbf{x})$ satisfying the equation $\operatorname{grad} V(\mathbf{x}) \cdot P\mathbf{x} = -\|\mathbf{x}\|^2$. Having made the substitution

$$\xi_s = e^{\alpha t} x_s, \quad \alpha = \text{const} > 0 \tag{2.4}$$

we transform the second group of equations in (2.1) to

$$\xi^{\boldsymbol{\cdot}} = (P + \alpha E)\xi + e^{\alpha t}\mathbf{X}(t, e^{-\alpha t}\xi, \mathbf{y}) \tag{2.5}$$

where E is the unit matrix. The time derivative of function $V(\xi)$, by virtue of (2.5), is

$$V^{\boldsymbol{\cdot}}(\xi) = -\|\xi\|^2 + 2\alpha V(\xi) + e^{\alpha t} \operatorname{grad} V(\xi)\, \mathbf{X}(t, e^{-\alpha t}\xi, \mathbf{y}) \tag{2.6}$$

We choose α so small that the form $-\|\xi\|^2 + 2\alpha V(\xi)$ is negative definite. By virtue of (1.3) there exists $\beta > 0$ such that the derivative (2.6) is nonpositive in the region

$$t \geqslant 0, \ \|\xi\| \leqslant \beta, \ \|\mathbf{y}\| \leqslant \beta \tag{2.7}$$

Let us consider an arbitrary solution $\xi(t, \xi_0, \mathbf{y}_0)$, $\mathbf{y}(t, \xi_0, \mathbf{y}_0)$ with initial perturbations (for $t = 0$) from the region

$$\|\xi_0\| < \delta, \quad \|\mathbf{y}_0\| < \delta, \quad \delta < \beta \tag{2.8}$$

For this solution the conditions

$$\|\xi(t, \xi_0, \mathbf{y}_0)\| \leqslant \beta, \quad \|\mathbf{y}(t, \xi_0, \mathbf{y}_0)\| \leqslant \beta \tag{2.9}$$

are satisfied at least on some interval $(0, T)$. For all $t \in (0, T)$ we have $V^{\boldsymbol{\cdot}}(\xi(t, \xi_0, \mathbf{y}_0)) \leqslant 0$ and, consequently, $V(\xi(t, \xi_0, \mathbf{y}_0)) \leqslant V(\xi_0)$. Since form V is positive definite, there follows the inequality

$$\|\xi(t, \xi_0, \mathbf{y}_0)\| < A, \quad t \in (0, T) \tag{2.10}$$

where A is arbitrarily small if δ is fairly small. By virtue of (2.4)

$$\|\mathbf{x}(t, \mathbf{x}_0, \mathbf{y}_0)\| < Ae^{-\alpha t}, \ t \in (0, T) \tag{2.11}$$

follows from (2.10) $(\mathbf{x}_0 = \xi_0)$. Condition (1.3) and inequality (2.11) yield the estimate

$$\|\mathbf{Y}(t, \mathbf{x}(t, \mathbf{x}_0, \mathbf{y}_0), \mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0))\| < AMe^{-\alpha t}, \quad t \in (0, T), \quad M = \text{const} \tag{2.12}$$

The function $\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0)$, is a solution of the system of linear inhomogeneous equations

$$\mathbf{y}^{\boldsymbol{\cdot}} = R\mathbf{y} + Q\mathbf{x}(t, \mathbf{x}_0, \mathbf{y}_0) + \mathbf{Y}(t, \mathbf{x}(t, \mathbf{x}_0, \mathbf{y}_0), \mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0)) \tag{2.13}$$

and can be represented by the Cauchy formula [3]

$$\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0) = U(t)\mathbf{y}_0 + \int_0^t U(t-\tau)[Q\mathbf{x}(\tau, \mathbf{x}_0, \mathbf{y}_0) + \mathbf{Y}(\tau, \mathbf{x}(\tau, \mathbf{x}_0, \mathbf{y}_0), \mathbf{y}(\tau, \mathbf{x}_0, \mathbf{y}_0))]\, d\tau \tag{2.14}$$

where $U(t)$ is the fundamental matrix of solutions of the linear system

$$\mathbf{y}^{\boldsymbol{\cdot}} = R\mathbf{y} \tag{2.15}$$

The stability of the zero solution of system (2.15) follows from the stability of the zero solution of system (2.2); but in such case [4]

$$\|U(t)\| \leqslant N = \text{const} \quad \text{for } t \geqslant 0 \tag{2.16}$$

On the basis of (2.11), (2.12) and (2.16), from (2.14) we obtain

$$\|\mathbf{y}(t, \mathbf{x}_0, \mathbf{y}_0)\| \leqslant N\|\mathbf{y}_0\| + (N\|Q\|A + NAM)\int_0^t e^{-\alpha\tau}\, d\tau = N\|\mathbf{y}_0\| + AN\alpha^{-1}(\|Q\| + M)(1 - e^{-\alpha t}) < N\|\mathbf{y}_0\| + AN\alpha^{-1}(\|Q\| + M) \tag{2.17}$$

Let ε be an arbitrarily small number and $0 < \varepsilon < \beta$. We choose the δ in (2.8) so small the the inequality $A < \varepsilon$ is satisfied and that the right-hand side of (2.17) is less than ε. From (2.10) and (2.17) it follows that the inequalities

$$\|\xi(t, \xi_0, y_0)\| < \varepsilon, \quad \|y(t, \xi_0, y_0)\| < \varepsilon \tag{2.18}$$

are satisfied throughout the time interval during which inequalities (2.9) are satisfied. But since $\varepsilon < \beta$, inequalities (2.18) are fulfilled for all $t \geqslant 0$. Consequently, the unperturbed motion is stable relative to ξ, y, whence the required result follows by virtue of (2.4).

Theorem 2. Assume that the zero solution of system (2.2) is exponentially-asymptotically x-stable and y-unstable. Then the unperturbed motion of system (2.1) is y-unstable for any functions Y, X satisfying condition (1.2).

Notes. 1) Condition (1.3) is not used in Theorem 2.

2) Under the hypotheses of Theorem 2 the y-instability of the zero solution of system (2.2) is equivalent to the existence in Eq. (2.3) of multiple roots with zero real parts and nonprime elementary divisors or of roots with a positive real part.

Proof. From the theorem's hypotheses it follows that the zero solution of system (2.15) is unstable. Consequently, system (2.15) has the solutions $y = y(t, y_0)$ with arbitrarily small $\|y_0\|$, leaving the region $\|y\| < H = \text{const}$ in the course of time. But then, by virtue of condition (1.2), system (2.1) has the solutions $(y = y(t, y_0)$, $x \equiv 0)$ with arbitrarily small initial conditions, which in the course of time leave the region $\|y\| < H$. The theorem is proved.

From Theorems 1 and 2 follows

Theorem 3. Let the equations of perturbed motion be of the form (2.1), the real parts of the roots of Eq. (1.4) be negative and the functions Y and X satisfy conditions (1.2) and (1.3). Then the question of the stability of the unperturbed motion of system (2.1) is completely resolved by the linear approximation (2.2): if the zero solution of system (2.2) is Liapunov-stable, the unperturbed motion of system (2.1) is Liapunov-stable and exponentially-asymptotically x-stable; if the zero solution of system (2.2) is y-unstable, the unperturbed motion of system (2.1) is y-unstable.

3. The results obtained are easily carried over to the stability problem with respect to a part of the variables [5] in the linear approximation [6]. Let the equations of perturbed motion have the form

$$\begin{gathered} y^{\cdot} = Qx + Ry + Y(t, x, y, z), \quad x^{\cdot} = Px + X(t, x, y, z) \\ z^{\cdot} = Z(t, x, y, z), \quad x \in R^n, \quad y \in R^k, \quad z \in R^r \end{gathered} \tag{3.1}$$

We assume that the solutions of system (3.1) are z-continuable and that conditions (3.3), (3.4) are satisfied in region (3.2)

$$t \geqslant 0, \quad \|x\| \leqslant h, \quad \|y\| \leqslant h, \quad \|z\| < \infty \tag{3.2}$$

$$Y(t, 0, y, z) \equiv 0, \quad X(t, 0, y, z) \equiv 0 \tag{3.3}$$

$$\frac{\|Y(t, x, y, z)\| + \|X(t, x, y, z)\|}{\|x\|} \underset{\substack{t \geqslant 0 \\ \|z\| < \infty}}{\rightrightarrows} 0 \quad \text{as } \|x\| + \|y\| \to 0 \tag{3.4}$$

Analogously to the preceding we can prove

Theorem 4. Let the equations of perturbed motion be of the form (3.1), the real parts of the roots of Eq. (1.4) be negative and the functions Y and X satisfy conditions (3.3) and (3.4) in region (3.2). Then the question of stability relative to (x, y) of the unperturbed motion of system (3.1) is completely resolved by the linear system (2.2):

if the zero solution of system (2.2) is Liapunov-stable, the unperturbed motion of system (3.1) is stable relative to $(\mathbf{x}, \mathbf{y})$ and exponentially-asymptotically $\mathbf{x}$-stable (in-the-large with respect to $\mathbf{z}_0$ [7]); if the zero solution of system (2.2) is $\mathbf{y}$-unstable, the unperturbed motion of system (3.1) is $\mathbf{y}$-unstable.

Note. If we know beforehand that the solutions of system (3.1) are $\mathbf{z}$-bounded uniformly with respect to $\{t_0, \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0\}$ [8], then Theorem 4 remains in force if conditions (3.3) and (3.4) are satisfied only in the region of the $\mathbf{z}$-boundedness [6].

4. Example 1. We consider a holonomic mechanical system with $n + k$ generalized coordinates $(q_1, \ldots, q_n, Q_1, \ldots, Q_k) = (\mathbf{q}, \mathbf{Q})$ and with time-independent couplings, whose kinetic energy T and quadratic part U_2 of the potential energy are

$$T = T^{(1)}(\mathbf{q}, \mathbf{Q}, \mathbf{q}^{\cdot}) + T^{(2)}(\mathbf{Q}^{\cdot}), \qquad U_2 = U_2^{(1)}(\mathbf{q}) + U_2^{(2)}(\mathbf{Q})$$

Here $U_2^{(1)}(\mathbf{q})$ is a positive-definite function. We assume that on the system there act forces gyroscopic with respect to $\mathbf{q}^{\cdot}$ and $\mathbf{Q}^{\cdot}$, dissipative forces with Rayleigh function $f(q_1^{\cdot}, \ldots, q_n^{\cdot})$ which is taken to be a positive-definite quadratic form in its arguments, as well as certain nonconservative forces of a higher than first order of smallness, so that the equations of motion of the system being examined have the form

$$\frac{d}{dt}\frac{\partial T^{(1)}}{\partial q_i^{\cdot}} - \frac{\partial T^{(1)}}{\partial q_i} = -\frac{\partial U_2^{(1)}}{\partial q_i} + \sum_{s=1}^{n} g_{is} q_s^{\cdot} - \frac{\partial f}{\partial q_i^{\cdot}} + F_i(\mathbf{q}, \mathbf{q}^{\cdot}, \mathbf{Q}, \mathbf{Q}^{\cdot}), \qquad i = 1, \ldots, n;\ g_{is} = -g_{si} \tag{4.1}$$

$$\frac{d}{dt}\frac{\partial T^{(2)}}{\partial Q_j^{\cdot}} - \frac{\partial T^{(1)}}{\partial Q_j} = -\frac{\partial U_2^{(2)}}{\partial Q_j} + \sum_{s=1}^{k} G_{js} Q_s^{\cdot} + \Phi_j(\mathbf{q}, \mathbf{q}^{\cdot}, \mathbf{Q}, \mathbf{Q}^{\cdot}), \qquad j = 1, \ldots, k;\ G_{js} = -G_{sj}$$

where the functions F_i and Φ_j do not contain linear terms and satisfy the condition

$$F_i(0, 0, \mathbf{Q}, \mathbf{Q}^{\cdot}) \equiv \Phi_j(0, 0, \mathbf{Q}, \mathbf{Q}^{\cdot}) \equiv 0 \tag{4.2}$$

In the linear-approximation equations for system (4.1) the variables $(\mathbf{q}, \mathbf{q}^{\cdot})$ and $(\mathbf{Q}, \mathbf{Q}^{\cdot})$ separate

$$\frac{d}{dt}\frac{\partial T_0^{(1)}}{\partial q_i^{\cdot}} = -\frac{\partial U_2^{(1)}}{\partial q_i} + \sum_{s=1}^{n} g_{is} q_s^{\cdot} - \frac{\partial f}{\partial q_i^{\cdot}}, \quad T_0^{(1)} = T^{(1)}(0, 0, \mathbf{q}^{\cdot}), \qquad i = 1, \ldots, n \tag{4.3}$$

$$\frac{d}{dt}\frac{\partial T^{(2)}}{\partial Q_j^{\cdot}} = -\frac{\partial U_2^{(2)}}{\partial Q_j} + \sum_{s=1}^{k} G_{js} Q_s^{\cdot}, \qquad j = 1, \ldots, k \tag{4.4}$$

the equilibrium position $\mathbf{q} = \mathbf{q}^{\cdot} = 0$ of system (4.3) is exponentially-asymptotically stable.

On the basis of Theorem 3 we conclude that if the form $U_2^{(2)}(\mathbf{Q})$ is positive definite, then the equilibrium position $q_i = q_i^{\cdot} = Q_j = Q_j^{\cdot} = 0$ of system (4.1) is Liapunov-stable and exponentially-asymptotically stable relative to $\mathbf{q}$, $\mathbf{q}^{\cdot}$; however, if $U_2^{(2)}(\mathbf{Q})$ takes negative values and all $G_{js} = 0$, then the equilibrium position $q_i = q_i^{\cdot} = Q_j = Q_j^{\cdot} = 0$ of system (4.1) is unstable relative to $\mathbf{Q}$, $\mathbf{Q}^{\cdot}$.

Notes. 1) This conclusion remains in force if the function f is sign-constant (the dissipation is partial with respect to $q_1^{\cdot}, \ldots, q_n^{\cdot}$), but [9] the equilibrium position $\mathbf{q} = \mathbf{q}^{\cdot} = 0$ of system (4.3) is asymptotically stable.

2) Condition (4.2) is satisfied for conservative forces if the system's potential energy has the form

$$U(\mathbf{q}, \mathbf{Q}) = U_2^{(1)}(\mathbf{q}) + U_2^{(2)}(\mathbf{Q}) + U'(\mathbf{q}, \mathbf{Q})$$

where U' are terms of order of smallness higher than second and $\partial U' / \partial q_i \equiv \partial U' / \partial Q_j \equiv 0$ for $\mathbf{q} = 0$.

Example 2. Let us consider a special case of Example 1. Let the system be reduced to normal coordinates [10] $x_1, \ldots, x_n$, $X_1, \ldots, X_k$, and let the equations of motion be

$$x_i^{\cdot\cdot} = -\lambda_i x_i + \sum_{s=1}^{n} g_{is} x_s^{\cdot} - \frac{\partial f}{\partial x_i^{\cdot}} + F_i(\mathbf{x}, \mathbf{x}^{\cdot}, \mathbf{X}, \mathbf{X}^{\cdot}), \quad i = 1, \ldots, n; \quad g_{is} = -g_{si} \quad (4.5)$$

$$X_j^{\cdot\cdot} = -\Lambda_j X_j + \sum_{s=1}^{k} G_{js} X_s^{\cdot} + \Phi_j(\mathbf{x}, \mathbf{x}^{\cdot}, \mathbf{X}, \mathbf{X}^{\cdot}), \quad j = 1, \ldots, k; \; G_{js} = -G_{sj}$$

where $f(x_1^{\cdot}, \ldots, x_n^{\cdot})$ is a positive-definite quadratic form. We assume that all $\lambda_i > 0$, while the functions F_i and Φ_j do not contain linear terms and satisfy the condition $F_i(0, 0, \mathbf{X}, \mathbf{X}^{\cdot}) \equiv \Phi_j(0, 0, \mathbf{X}, \mathbf{X}^{\cdot}) \equiv 0$. On the basis of Theorem 3 we conclude that if all $\Lambda_j > 0$, then the equilibrium position $x_i = x_i^{\cdot} = X_j = X_j^{\cdot} = 0$ of system (4.5) is Liapunov-stable and exponentially-asymptotically $(\mathbf{x}, \mathbf{x}^{\cdot})$-stable; if $\Lambda_j \leqslant 0$ exists and all $G_{js} = 0$, then the equilibrium position $x_i = x_i^{\cdot} = X_j = X_j^{\cdot} = 0$ of system (4.5) is unstable relative to $\mathbf{X}$, $\mathbf{X}^{\cdot}$.

Example 3. Example 1 can be extended to the stability problem for steadystate motions [11]. Let us assume that for the given values of the cyclic integrals $p_\alpha = c_\alpha$ the position coordinates q_j $(j = 1, \ldots, k)$ separate into two groups: $q_1, \ldots, q_m$ and $q_{m+1}, \ldots, q_k$; the part R_2, quadratic relative to the positional velocities of the Routh function R is of form (4.6), while the potential energy of the reduced system is of form (4.7)

$$R_2 = R_2^{(1)}(q_1, \ldots, q_k, q_1^{\cdot}, \ldots, q_m^{\cdot}) + R_2^{(2)}(q_{m+1}^{\cdot}, \ldots, q_k^{\cdot}) \quad (4.6)$$

$$W = W_2^{(1)}(q_1, \ldots, q_m) + W_2^{(2)}(q_{m+1}, \ldots, q_k) + W'(q_1, \ldots, q_k) \quad (4.7)$$

Here $W_2^{(1)}$ and $W_2^{(2)}$ are quadratic forms in their arguments, the first of which is positive definite, W' are terms of higher than second order of smallness, $\partial W' / \partial q_i \equiv 0$ $(i = 1, \ldots, k)$ for $q_1 = \ldots = q_m = 0$.

Further, on the system let there act dissipative forces with Rayleigh function $f(q_1^{\cdot}, \ldots, q_m^{\cdot})$ and certain nonconservative forces of higher than second order of smallness, not having values p_α, while the gyroscopic forces that arise due to the presence of terms linear in $q_j^{\cdot}$ in the Routh function, are such that the equations of motion [11] take the form

$$\frac{d}{dt}\frac{\partial R_2^{(1)}}{\partial q_i^{\cdot}} - \frac{\partial R_2^{(1)}}{\partial q_i} = -\frac{\partial W_2^{(1)}}{\partial q_i} + \sum_{s=1}^{m} g_{is} q_s^{\cdot} - \frac{\partial f}{\partial q_i^{\cdot}} + F_i, \quad i = 1, \ldots, m \quad (4.8)$$

$$\frac{d}{dt}\frac{\partial R_2^{(2)}}{\partial q_i^{\cdot}} - \frac{\partial R_2^{(1)}}{\partial q_i} = -\frac{\partial W_2^{(2)}}{\partial q_i} + \sum_{s=m+1}^{k} g_{is} q_s^{\cdot} + F_i, \quad i = m+1, \ldots, k$$

Here the functions F_i $(i = 1, \ldots, k)$ do not contain linear terms and $F_i \equiv 0$ $(i = 1, \ldots, k)$ for $q_1 = q_1^{\cdot} = \ldots = q_m = q_m^{\cdot} = 0$.

On the basis of Theorem 3 we conclude that if form $W_2^{(2)}$ is positive definite, then the steadystate motion $(p_\alpha = c_\alpha,\ q_j = q_j^{\cdot} = 0)$ is stable relative to $q_1, \ldots, q_k, q_1^{\cdot}, \ldots, q_k^{\cdot}$ and exponentially-asymptotically stable relative to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_m^{\cdot}$ at least for perturbations not having the values $p_\alpha = c_\alpha$; if $W_2^{(2)}$ takes negative values, while gyroscopic terms are absent in the second group of Eqs. (4. 8), then the given steady-state motion is unstable relative to $q_{m+1}, \ldots, q_k, q_{m+1}^{\cdot}, \ldots, q_k^{\cdot}$.

Example 4. Let us consider the connection between the results obtained in Sect. 2 and the critical cases of one pair and of several pairs of pure imaginary roots. Let the equations of perturbed motion have the form

$$\begin{aligned} x_s^{\cdot} &= p_{s1}x_1 + \ldots + p_{sn}x_n + X_s(x, y, x_1, \ldots, x_n), \quad s = 1, \ldots, n \\ x^{\cdot} &= q_{11}x_1 + \ldots + q_{1n}x_n + p_1 x + q_1 y + X(x, y, x_1, \ldots, x_n) \\ y^{\cdot} &= q_{21}x_1 + \ldots + q_{2n}x_n + p_2 x + q_2 y + Y(x, y, x_1, \ldots, x_n) \end{aligned} \qquad (4.9)$$

and let the roots of Eq. (1.4) have negative real parts, the equation

$$\begin{vmatrix} p_1 - \lambda & q_1 \\ p_2 & q_2 - \lambda \end{vmatrix} = 0$$

have a pair of pure imaginary roots, and X, Y and X_s be analytic functions whose expansions commence with terms of no lower than second order and which satisfy the condition

$$X(x, y, 0, \ldots, 0) \equiv Y(x, y, 0, \ldots, 0) \equiv X_s(x, y, 0, \ldots, 0) \equiv 0 \qquad (4.10)$$

From Theorem 1 it follows that the unperturbed motion of system (4. 9) is Liapunov-stable and exponentially-asymptotically stable relative to $x_1, \ldots, x_n$. By virtue of (4. 10) the stability with respect to x, y is not asymptotic (it is sufficient to set $x_{s0} = 0$, $x_0^2 + y_0^2 \neq 0$); consequently, we have to deal with a special case [2] of the critical case of a pair of pure imaginary roots.

Analogous conclusions are valid in the critical case [12] of several pairs of pure imaginary roots independently of the presence of resonance relations between the imaginary roots.

The author thanks V. V. Rumiantsev for attention to this paper.

REFERENCES

1. Malkin, I. G., On the stability of motion in the sense of Liapunov. Matem. Sb., Vol. 3, №1, 1938.
2. Malkin, I. G., Theory of Stability of Motion. Moscow, "Nauka", 1966.
3. Krasovskii, N. N., Theory of Control of Motion. Moscow, "Nauka", 1968.
4. Demidovich, B. P., Lectures on Mathematical Stability Theory. Moscow, "Nauka", 1967.
5. Oziraner, A. S. and Rumiantsev, V. V., The method of Liapunov functions in the stability problem for motion with respect to a part of the variables. PMM Vol. 36, №2, 1972.
6. Oziraner, A. S., On asymptotic stability and instability relative to a part of variables. PMM Vol. 37, №4, 1973.
7. Rumiantsev, V. V., Certain problems on the stability of motion relative to a part of the variables. In: Mechanics of a Continuous Medium and Related Problems of Analysis. Moscow, "Nauka", 1972.
8. Oziraner, A. S., On certain theorems of Liapunov's second method. PMM Vol. 36, №3, 1972.
9. Pozharitskii, G. K., On the asymptotic stability of the equilibria and the steady-state motions of mechanical systems with partial dissipation. PMM Vol. 25, №4, 1961.

10. Chetayev, N. G., The Stability of Motion. (English translation), Pergamon Press, Book № 09505, 1961.
11. Rumiantsev, V. V., On the Stability of the Steadystate Motions of Satellites. Moscow, Izd.Vychisl. Tsentr. Akad. Nauk SSSR, 1967.
12. Zubov, V. I., Stability of Motion. Moscow, "Vysshaia Shkola", 1973.

Translated by N. H. C.

UDC 531.36

ON STABILITY OF MOTION WITH RESPECT TO A PART OF VARIABLES IN THE CRITICAL CASE OF A SINGLE ZERO ROOT

PMM Vol. 39, № 3, 1975, pp. 422-426
V. P. PROKOP'EV
(Sverdlovsk)
(Received July 8, 1974)

The stability of motion with respect to a part of variables is investigated for the critical case of a single zero root. The criterions of stability and instability are obtained.

1. Consider the system of differential equations of perturbed motion

$$dx_i/dt = X_i(x_1 \ldots, x_n), \qquad i = 1, \ldots, n \tag{1.1}$$

We shall investigate the problem of stability of unperturbed motion $x_i = 0$ $(i = 1, \ldots, n)$ with respect to $x_1, \ldots, x_m$ $(m > 0, \quad n = m + p, \quad p > 0)$. We denote these variables by $y_i = x_i$ $(i = 1, \ldots, m)$, and the remaining ones by $z_j = x_{m+j}$ $(j = 1, \ldots, p)$ [1, 2]. Let the functions X_i represent power series expanded in the powers of y_i $(i = 1, \ldots, m)$ and z_j $(j = 1, \ldots, p)$ and convergent in the region

$$|y_i| \leqslant h, \quad i = 1, \ldots, m, \quad |z_j| \leqslant H < \infty, \quad j = 1, \ldots, p \tag{1.2}$$

where h and H are certain constants.

Now the equations of perturbed motion (1.1) assume the form

$$\frac{dy_i}{dt} = \sum_{j=1}^{m} a_{ij} y_j + Y_i(y_1, \ldots, y_m, z_1, \ldots, z_p), \quad i = 1, \ldots, m \tag{1.3}$$

$$\frac{dz_k}{dt} = \sum_{j=1}^{m} b_{kj} y_j + \sum_{j=1}^{p} c_{kj} z_j + Z_k(y_1, \ldots, y_m, z_1, \ldots, z_p), \quad k = 1, \ldots, p$$

where a_{ij}, b_{kj} and c_{kj} are constants, Y_i and Z_k are functions of the variables $y_1, \ldots, y_m, z_1, \ldots, z_p$ which are expanded in the region (1.2) into power series in these variables with the first terms of at least second order. The variables z_j $(j = 1, \ldots, p)$ are always bounded (Condition A), since they belong to the region (1.2). This condition represents the starting assumption in the investigation of (1.3).

Let

$$Y_i(0, \ldots, 0, z_1, \ldots, z_p) = 0, \quad |Y_i| \leqslant \sum_{j=1}^{m} h_{ij} |y_j|, \quad i = 1, \ldots, m \tag{1.4}$$